Seat No. : _______________

ZO-128

May-2014

M.Sc. Sem.-II

Mathematics (Algebra – I)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**

   (1) Prove that every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

   (2) Prove that $\text{Aut}(\mathbb{Z}_n)$ is isomorphic to $U(n)$.

   (b) Attempt any **two** : **4**

   (1) Let $\alpha = (1, 3, 5, 7, 9, 8, 6)\ (2, 4, 10)$. What is the smallest positive integer n for which $\alpha^{-4} = \alpha^n$?

   (2) Prove that $(\mathbb{Z}, +)$ is not isomorphic to $(Q, +)$.

   (3) If G is an Abelian group of odd order, prove that the product of all the elements of G is the identity.

   (c) Attempt **all** : **3**

   (1) Give an example of a group G with a proper subgroup H such that G and H are isomorphic.

   (2) Give an example of (non-identity) isomorphism from $(\not\subset +)$ to $(\not\subset +)$.

   (3) Give an example of an infinite group in which the order of each element is finite.

2. (a) Attempt any **one** : **7**

   (1) Let G be a group and let Z(G) be the centre of G. If $\frac{G}{Z(G)}$ is cyclic, prove that G is Abelian.

   (2) Let G and H be finite cyclic groups. Then prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

   (b) Attempt any **two** : **4**

   (1) What are the last two digits of 17^{102} ?

   (2) Find a subgroup of $\mathbb{Z}_{12} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{15}$ that is of order 9.

   (3) If G is an non-Abelian group of order 21 then show that $Z(G) = (e)$.

(c) Attempt **all** : **3**

(1) Prove or disprove : U(5) is isomorphic to U(8).

(2) Let G be a group and H = $\{g^2/g \in G\}$ is a subgroup of G, then prove that H is normal in G.

(3) Define Direct Product.

3. (a) Attempt any **one** : **7**

(1) State (only) fundamental theorem of finite Abelian groups. Determine the isomorphism class of group U(20).

(2) State and prove first Isomorphism theorem.

(b) Attempt any **two** : **4**

(1) Let H be a normal subgroup of a finite group G. If gcd $(|x|, |G/H|) = 1$ then show that $x \in$ H.

(2) Prove that $\phi : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by ϕ (a, b) = a – b is a homomorphism. Find the Kernal of ϕ. Find $\phi^{-1}(3)$.

(3) Prove that every Abelian group of order 45 has an element of order 15. Does every Abelian group of order 45 have an element of order 9 ? Justify your answer.

(c) Attempt **all** : **3**

(1) How many (non-isomorphic) Abelian groups of order 196 are there ? Why ?

(2) Prove that any Abelian group of order pq (p, q are primes, $p \neq q$) is cyclic.

(3) How many homomorphism are there from (Q, +) to (Q, +) ?

4. (a) Attempt any **one** : **7**

(1) State and prove Sylow's first theorem.

(2) State and prove Cauchy's theorem.

(b) Attempt any **two** : **4**

(1) Prove that any group of order 99 is Abelian.

(2) Prove that any two p-Sylow subgroups of a group G are isomorphic.

(3) Prove that a non-cyclic group of order 21 must have 14 elements of order 3.

(c) Attempt **all** : **3**

(1) Define Contugacy relation. Prove that this relation is an equivalence relation.

(2) What is the smallest possible odd integer that can be the order of a non-Abelian group ? Explain briefly.

(3) Prove that a group of order 100 has a subgroup of order 5.

5. (a) Attempt any **one** : **7**

(1) State and prove Sylow theorem for non-simplicity.

(2) Prove that A_5 is simple

(b) Attempt any **two** : **4**

(1) Prove that there is no simple group of order 80.

(2) If H is a subgroup of S_5 and H contains a 2-cycle and a 5-cycle, prove that $H = S_5$.

(3) Show that A_5 can not contain a sub group of order 30.

(c) Attempt **all** : **3**

(1) Prove that any group of prime order is simple.

(2) State (only) Burnside theorem.

(3) Determine the number of ways in which the vertices of an equilateral triangle can be coloured with five colours so that at least two colours are used.
